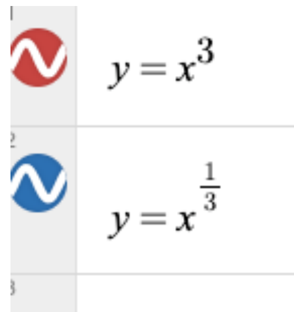
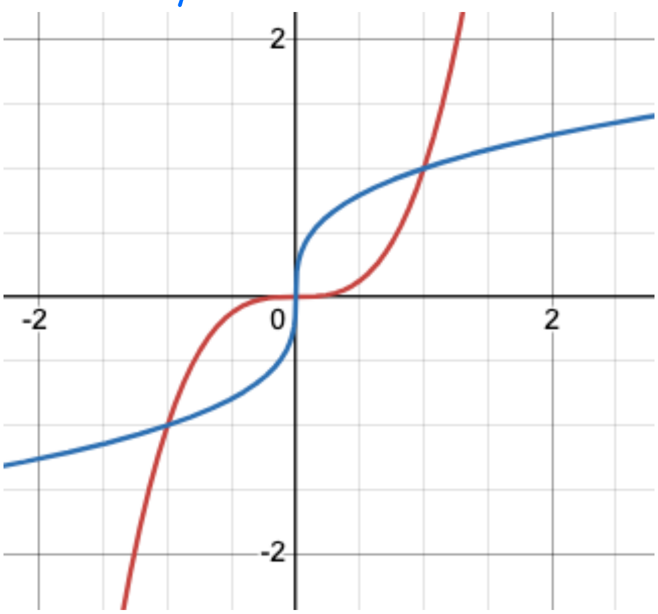
Period 3, Oct 25, 2024

P350 #81, 83   379 # 49, 53, 65, 67, 51

81. $F(x)=x^3$ $\left(\frac{1}{2}, \frac{1}{8}\right)$

$F^{-1}(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ $\left(\frac{1}{8}, \frac{1}{2}\right)$





$F'(x)=3x^2$

$(F^{-1})'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

$F'\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 = 3\cdot\frac{1}{4} = \frac{3}{4}$

$(F^{-1})'\left(\frac{1}{8}\right) = \frac{1}{3\sqrt[3]{\left(\frac{1}{8}\right)^2}} = \frac{1}{3\sqrt[3]{\frac{1}{64}}} = \frac{1}{3\cdot\frac{1}{4}}$

$\frac{4}{3} = \frac{1}{\frac{3}{4}}$

83. $x \geq 4$ domain

$F(x)=\sqrt{x-4}=(x-4)^{\frac{1}{2}}$ Range $y \geq 0$

$F'(x) = \frac{1}{2\sqrt{x-4}}$

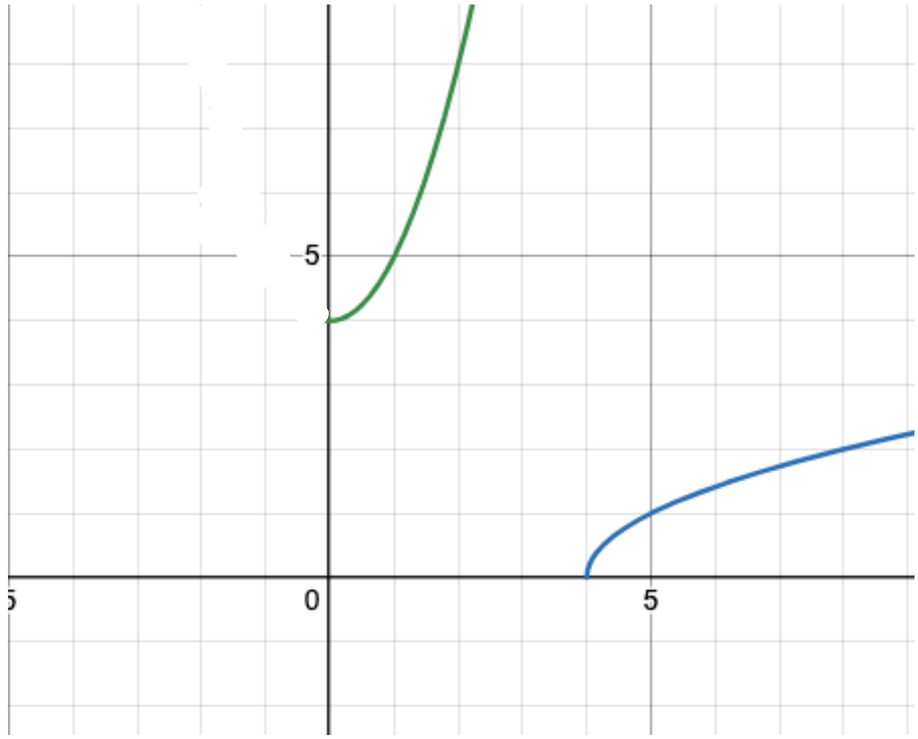
$F^{-1}(x) = x^2+4$ Domain $x \geq 0$ Range $y \geq 4$

$(F^{-1})'(x) = 2x$

$(5,1)$ $F'(5) = \frac{1}{2\sqrt{5-4}}$

$(1,5)$ $F'(5) = \frac{1}{2}$

$(F^{-1})'(1) = 2(1) = 2$



379 # 49, 53, 65, 67, 51

49. $g(x) = \frac{\arcsin 3x}{x} \Rightarrow g'(x) = \frac{\frac{3}{\sqrt{1-(3x)^2}} \cdot x - (\arcsin 3x) \cdot 1}{x^2}$

$u = 3x$

$u' = 3$

$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$

$g'(x) = \frac{\frac{3x}{\sqrt{1-9x^2}} - \frac{(\sqrt{1-9x^2}) \arcsin 3x}{\sqrt{1-9x^2}}}{\frac{x^2}{1}}$

$g'(x) = \frac{3x - \sqrt{1-9x^2}\ \arcsin 3x}{\sqrt{1-9x^2}} \cdot \frac{1}{x^2}$

51. $h(T) = \sin(\arccos T)$

$y = \sin(\arccos T) \Rightarrow y = \sin u$

$u = \arccos T$ $\qquad \frac{dy}{du} = \cos u$

$\frac{du}{dx} = \frac{-1}{\sqrt{1-T^2}}$

$$\frac{du}{dx} \cdot \frac{dy}{du} = \frac{dy}{dx} = \frac{-1}{\sqrt{1-T^2}} \cdot \overbrace{\cos(\arccos T)}^{T}$$

$$= \frac{-T}{\sqrt{1-T^2}}$$

53.

$$y = \underbrace{2x}\,\underbrace{\arccos x} - 2\sqrt{1-x^2} = 2x\arccos x - 2(1-x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2 \cdot \arccos x + 2x \cdot \frac{-1}{\sqrt{1-x^2}} - \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot \frac{1}{\sqrt{1-x^2}} \cdot -2x$$

$$2\arccos x \cancel{- \frac{2x}{\sqrt{1-x^2}}} \cancel{+ \frac{2x}{\sqrt{1-x^2}}}$$

65. $y = \arctan\frac{x}{2}$

$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$

$u = \frac{x}{2}$

$u' = \frac{1}{2}$

$(2, \frac{\pi}{4})$
$x_1 \ y_1$

plug in $x=2$ into $\frac{dy}{dx}$ for slope of Tangent Line

$\frac{dy}{dx} = \frac{\frac{1}{2}}{1+(\frac{x}{2})^2} =$

$\frac{dy}{dx} = \frac{\frac{1}{2}}{1+(\frac{2}{2})^2} = \frac{\frac{1}{2}}{2} = \frac{1}{4} = m$

Line $y - y_1 = m(x - x_1)$

$y - \frac{\pi}{4} = \frac{1}{4}(x-2)$

$y - \frac{\pi}{4} = \frac{1}{4}x - \frac{1}{2}$
$+\frac{\pi}{4} \qquad +\frac{\pi}{4}$

$\Rightarrow \boxed{y = \frac{1}{4}x - \frac{1}{2} + \frac{\pi}{4}}$

67.

$y = 4x\arccos(x-1)$

$\frac{d}{dx}[\arccos(x-1)]$
$u = x-1$

$= \frac{-1}{\sqrt{1-(x-1)^2}} = \frac{-1}{\sqrt{\cancel{x}-x^2+2x-\cancel{1}}}$

$= \frac{-1}{\sqrt{2x-x^2}}$

$\frac{dy}{dx} = 4\cdot\arccos(x-1) + 4x\cdot\frac{-1}{\sqrt{2x-x^2}}$

$= 4\arccos(x-1) - \frac{4x}{\sqrt{2x-x^2}}$

$(1, 2\pi)$
Point

$4\arccos(\overset{0}{1-1}) - \frac{4(1)}{\sqrt{2-1}} = 4\cdot\frac{\pi}{2} - \frac{4}{1} = 2\pi - 4 =$ Slope $= m$

$\arccos 0 = p$

$\cos p = 0$

$\cos \frac{\pi}{2} = 0$

$p = \frac{\pi}{2}$

$y - 2\pi = (2\pi - 4)(x - 1)$

$y - \cancel{2\pi} = 2\pi x - \cancel{2\pi} - 4x + 4$

$+2\pi \qquad +2\pi$

$y = x(2\pi - 4) + 4$

73.

$F(x) = x^3 + 2x - 1 \qquad a = 2$

$F'(x) = 3x^2 + 2$

$F'(1) = 3(1)^2 + 2 = 5$

$F^{-1}(2) = k \quad \Rightarrow (2, k)$

Try

$F(k) = 2 \quad \Rightarrow \quad 2 = k^3 + 2k - 1$

$k = \pm 1, \pm 2, \pm 3$

Try $k = 1$

$1^3 + 2(1) - 1 = \cancel{1} + 2 - \cancel{1} = 2$

$k = 1$

$$(F^{-1})'(2) = \frac{1}{F'(F^{-1}(2))} = \frac{1}{F'(1)} = \frac{1}{5}$$

75.

$F(x)=\sin x$

$\frac{1}{2}=\sin x$

$x=\frac{\pi}{6}$

$F'(x)=\cos x$

$a=\frac{1}{2}$

$F(\frac{\pi}{6})=\frac{1}{2}$

$F^{-1}(\frac{1}{2})=\frac{\pi}{6}$

$$(F^{-1})'(\tfrac{1}{2})=\frac{1}{F'(F^{-1}(\frac{1}{2}))}$$

$$(F^{-1})'(\tfrac{1}{2})=\frac{1}{\cos\frac{\pi}{6}}=\frac{1}{\frac{\sqrt{3}}{2}}$$

$$(F^{-1})'(\tfrac{1}{2})=\frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

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e) $f(x)=\underbrace{x\arcsin(1-x^2)}_{uv}$

$$\frac{d}{dx}\left[\arcsin u\right]=\frac{u'}{\sqrt{1-u^2}}$$

$u=1-x^2 \Rightarrow u'=-2x$

$$F'(x)=1\cdot\arcsin(1-x^2)+x\cdot\frac{-2x}{\sqrt{1-(1-x^2)^2}}$$

$$F'(x)=\arcsin(1-x^2)-\frac{2x^2}{\sqrt{\cancel{1-1}+2x^2-x^4}}=$$

$$\arcsin(1-x^2)-\frac{2x^2}{\sqrt{2x^2-x^4}}=\arcsin(1-x^2)-\frac{2x}{|x|\sqrt{2-x^2}}$$

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6. Find the points where the graph of $25x^2 + 16y^2 + 200x - 160y + 400 = 0$ has a horizontal and vertical tangent lines.

$$\frac{dy}{dx} = \frac{a}{0} \qquad \frac{dy}{dx} = 0$$

$$50x + 32y\frac{dy}{dx} + 200 - 160\frac{dy}{dx} = 0$$

$$50x + 200 = \frac{dy}{dx}(-32y + 160)$$

$$\frac{50x+200}{(160-32y)} = \frac{dy}{dx}$$

$50x + 200 = 0$
$x = -4$
Horizontal Tangent

$160 - 32y = 0$
$y = 5$
Vertical Tangent

$$25(-4)^2 + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$\cancel{400} + 16y^2 - \cancel{800} - 160y + \cancel{400} = 0$$

$$16y(y-10) \qquad (-4, 0)$$

$$x = -4, \quad y = 0 \text{ or } 10 \qquad (-4, 10)$$

Veritical Tangent

$25x^2 + 16(5)^2 + 200x - 160(5) + 400$

Solve For x

4. Find the equation of the tangent line at the indicated point

a) $(y - x)^2 + y^3 = xy + 7$ at $(1, 2)$

b) $\frac{x^2}{16} + y^2$

$$2(y-x)\left(\frac{dy}{dx} - 1\right) + 3y^2\frac{dy}{dx} = 1\cdot y + x\cdot\frac{dy}{dx}$$

$$2(2-1)\left(\frac{dy}{dx} - 1\right) + 3(2)^2\frac{dy}{dx} = 1\cdot 2 + 1\cdot\frac{dy}{dx}$$

keep going

5. Find the equation of the normal line at the indicated point

a) $y^3x + 2y = x^2$ at $(2,1)$

$$3y^2\frac{dy}{dx}\cdot x + y^3\cdot 1 + 2\frac{dy}{dx} = 2x$$

$$3(1)^2\frac{dy}{dx}\cdot 2 + 1^3 + 2\frac{dy}{dx} = 2(2)$$

$$6\frac{dy}{dx} + 1 + 2\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{3}{8}$$

keep going

b) $y\sqrt{x} - x\sqrt{y} = 12$ at $(9,16)$

$$y\cdot x^{\frac{1}{2}} - x\cdot y^{\frac{1}{2}} = 12$$

$$\frac{dy}{dx}\cdot x^{\frac{1}{2}} + y\cdot\frac{1}{2\sqrt{x}} - 1\sqrt{y} - x\cdot\frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\sqrt{9} + \frac{16}{2\sqrt{9}} - \sqrt{16} - \frac{9}{2\sqrt{16}}\frac{dy}{dx} = 0$$

$$\left(3 - \frac{9}{8}\right)\frac{dy}{dx} = 4 - \frac{16}{6} = \frac{8}{6}$$

$$\frac{15}{8}\frac{dy}{dx} = \frac{8}{6}$$

$$\frac{dy}{dx} = \frac{64}{90} = \frac{32}{45} = \text{Tangent Slope}$$

$$\text{Normal slope} = \frac{-45}{32}$$

Point $= (9,16)$

$m = \frac{-45}{32}$

$$y - 16 = -\frac{45}{32}(x-9)$$

$\log_b a = c \iff b^c = a$ $\qquad$ $\ln_e a = c \iff e^c = a$

$\log_b a + \log_b c = \log_b ac$ $\qquad$ $\ln a + \ln c = \ln ac$

$\log_b a - \log_b c = \log_b \frac{a}{c}$ $\qquad$ $\ln a - \ln c = \ln \frac{a}{c}$

$\log_b a^c = c \log_b a$ $\qquad$ $\ln a^c = c \ln a$

$\ln e = 1$

The Derivative of the Natural Logarithmic Function

Derivative of $\ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

More Generally....

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \qquad , u > 0$$

$$= \frac{u'}{u}$$

1. Find y' if $y = \ln(2x + 2)$

$u = 2x+2 \Rightarrow u' = 2$

$\frac{dy}{dx} = y' = \frac{1}{2x+2} \cdot 2 = \frac{\cancel{2}}{\cancel{2}(x+1)} = \frac{1}{x+1}$

2. Let $f(x) = \ln(\tan x)$. Find $f'(x)$

$u = \tan x \Rightarrow u' = \sec^2 x$

$\frac{1 \cdot \cancel{\cos x}}{\cancel{(\cos x)}(\cos x)(\sin x)}$

$F'(x) = \frac{\sec^2 x}{\tan x} = \frac{1}{\cos^{\cancel{2}} x} \cdot \frac{\cancel{\cos x}}{\sin x} = \sec x \cdot \csc x$

3. $\frac{d}{dx}[\ln(x^2 + 1)] \Rightarrow \frac{dy}{dx} = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

$u = x^2+1$

$u' = 2x$

Product Rule

4. $\frac{d}{dx}[x \ln x]$

$F'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= \ln x + \frac{x}{x} = \ln x + 1$

Chain Rule

5. $\frac{d}{dx}\left[(\ln x)^3\right]$

$u = \ln x$ $\qquad y = u^3$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad \frac{dy}{du} = 3u^2$

$\frac{du}{dx} \cdot \frac{dy}{du} = \frac{1}{x} \cdot 3u^2 = \frac{3(\ln x)^2}{x}$

Example Set B: Expanding Logarithmic Expressions

1) $\ln\left(\frac{10}{9}\right) = \ln 10 - \ln 9$

2) $\ln\sqrt{3x+2} = \ln(3x+2)^{\frac{1}{2}} = \frac{1}{2}\ln(3x+2)$

3) $\ln\frac{6x}{5} = \ln 6x - \ln 5 = \ln 6 + \ln x - \ln 5$

Example Set C

1. $\frac{d}{dx}\left[\ln\sqrt{x+1}\right] = \frac{d}{dx}\left[\ln(x+1)^{\frac{1}{2}}\right] = \frac{d}{dx}\left[\frac{1}{2}\ln(x+1)\right]$

$$\frac{1}{2}\cdot\frac{1}{x+1}\cdot 1 = \frac{1}{2(x+1)}$$

2. $\frac{d}{dx}\left(\ln\frac{x\left(x^2+1\right)^2}{\sqrt{2x^3-1}}\right)$

$$\frac{d}{dx}\left(\ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)\right)$$

$$= \frac{1}{x} + \frac{2}{x^2+1}\cdot 2x - \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot 6x^2$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3\cancel{6}x^2}{\cancel{2}(2x^3-1)}$$

Change of Base	$log_b a = \frac{\ln a}{\ln b}$	$log_2 3 = \frac{\ln 3}{\ln 2}$

$$\boldsymbol{log_a x = \frac{1}{\ln a}\ln x}$$

Change of base formula

1. Re write $f(x)$ using the properties of logs and find $f'(x)$

$$f(x) = \log_5 \sqrt{x} = \log_5 x^{\frac{1}{2}} = \frac{1}{2}\log_5 x = \underbrace{\frac{1}{2} \cdot \frac{1}{\ln 5}}_{\text{constant}} \cdot \ln x$$

$$F'(x) = \frac{1}{2\ln 5} \cdot \frac{1}{x} = \frac{1}{(\ln 5^2)x} = \frac{1}{(\ln 25)x} = \frac{1}{\ln 25^x}$$

Derivatives for Log Functions of Base a

Let $\boldsymbol{a}$ be a positive real number $(\boldsymbol{a} \neq \mathbf{1})$ and let $\boldsymbol{u}$ be a differentiable function of $\boldsymbol{x}$

$$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$$

More Generally….

$$\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$$

$$y = \log_a x \Leftrightarrow a^y = x$$

$$\ln a^y = \ln x$$

$$\frac{y \cancel{\ln a}}{\cancel{\ln a}} = \frac{\ln x}{\ln a}$$

$$y = \underbrace{\frac{1}{\ln a}}_{\text{constant}} \cdot \ln x$$

$$\frac{dy}{dx} = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

Example Set E #1

$$y = \log_7 x \Rightarrow \frac{dy}{dx} = \frac{1}{x \ln 7}$$

Example Set E #2

$$y = \log_7(5x^3 + 6)$$

$$\frac{dy}{dx} = \frac{1}{(\ln 7)(5x^3+6)} \cdot 15x^2 = \frac{15x^2}{(5x^3+6)(\ln 7)}$$

Example Set E #3

$$y = \log_7(\tan(\ln x^2))$$

$y = \tan(\ln x^2)$

$u = x^2$ $\quad L = \ln u$ $\quad y = \tan L$

$\frac{du}{dx} = 2x$ $\quad \frac{dL}{du} = \frac{1}{u}$ $\quad \frac{dy}{dL} = \sec^2 L$

$$\frac{dy}{dx} = \frac{1}{(\ln 7)(\tan(\ln x^2))} \cdot \left(2x \cdot \frac{1}{x^2} \cdot \sec^2(\tan(\ln(x^2)))\right)$$

$\ln a^b = b \cdot \ln a$

1. $\frac{d}{dx}[2^x]$

$y = 2^x$

$\ln y = \ln 2^x = x \overbrace{\ln 2}^{\text{constant}}$

$y \cdot \frac{1}{y}\frac{dy}{dx} = \ln 2 \cdot y$

$\frac{dy}{dx} = (\ln 2)y = (\ln 2)2^x$

Derivatives of a^x
Let a be a constant $\frac{d}{dx}[a^x] = \ln a \cdot a^x$

More Generally....

$$\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$$

$$= (\ln a)a^u \cdot u'$$

Example Set F: Find the Derivative

3. $y = 7^{\sin(2-\pi x)}$

4. $y = 7^{\csc(x^2)}$

$\ln y = \ln 7^{\sin(2-\pi x)}$

$\ln y = (\sin(2-\pi x))\overbrace{\ln 7}^{\text{constant}}$

$y \cdot \frac{dy}{dx} \cdot \frac{1}{\cancel{y}} = (\ln 7)(\cos(2-\pi x)) \cdot -\pi \cdot y$

$\frac{dy}{dx} = (\ln 7)(\cos(2-\pi x)) \cdot (-\pi)(7^{\sin(2-\pi x)})$

$y = 7^{\csc x^2}$

$\ln y = \ln 7^{\csc x^2} = \csc x^2 \cdot \overbrace{\ln 7}^{\text{constant}}$

$\cancel{y} \cdot \frac{1}{\cancel{y}} \frac{dy}{dx} = (\ln 7)(-\csc x^2 \cot x^2) \cdot 2x \cdot y$

$\frac{dy}{dx} = (\ln 7)(-\csc x^2 \cot x^2) \cdot 2x \cdot 7^{\csc x^2}$

Find the derivative of $f(x) = e^{5x} + 7^{2x} + \ln(x^2 + 4)$

$$F'(x) = e^{5x} \cdot 5 + (\ln 7)\, 7^{2x} \cdot 2 + \frac{1}{x^2+4} \cdot 2x$$

1. Find the y' if $y = x^x$

$$\ln y = \ln x^x = x \ln x$$

$$\cancel{y} \cdot \frac{1}{\cancel{y}} \frac{dy}{dx} = \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \cdot y$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

$$y = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$$

$$\ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1)$$

$$\cancel{y}\cdot\frac{1}{\cancel{y}}\frac{dy}{dx} = \left(\frac{1}{x} + \frac{1}{\cancel{2}}\cdot\frac{1}{x^2+1}\cdot \cancel{2}x - \frac{2}{3}\cdot\frac{1}{x+1}\cdot 1\right)\cdot y$$

$$\frac{dy}{dx} = \left(\frac{1}{x} + \frac{x}{(x^2+1)} - \frac{2}{3(x+1)}\right)\left(\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\right)$$

Your Turn #2

$$y = \sqrt[3]{\frac{(2x+3)^2(x-2)^4}{(x+1)}}$$

$$\ln y = \ln\left[\frac{(2x+3)^2(x-2)^4}{(x+1)}\right]^{\frac{1}{3}}$$

$$\ln y = \frac{1}{3}\left[2\ln(2x+3) + 4\ln(x-2) - \ln(x+1)\right]$$

$$\cancel{y}\cdot\frac{1}{\cancel{y}}\frac{dy}{dx} = \frac{1}{3}\left[2\cdot\frac{1}{2x+3}\cdot 2 + 4\cdot\frac{1}{x-2}\cdot 1 - \frac{1}{x+1}\cdot 1\right]\cdot y$$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{4}{2x+3} + \frac{4}{x-2} - \frac{1}{x+1}\right]\cdot\sqrt[3]{\frac{(2x+3)^2(x-2)^4}{(x+1)}}$$